

Math 336 Sample Problems

One notebook sized page of notes (both sides) will be allowed on the test. The test will be comprehensive. The final exam is at 8:30 am, Monday, June 4, in the regular classroom.

1. If f is analytic and nonconstant on $\mathbb{C} \setminus \{z_1, z_2, \dots, z_n\}$ then the image of f is dense in \mathbb{C} .
2. Let $p(z)$ be a polynomial of degree n . Let $M(r) = \max\{|p(z)| : |z| = r\}$. Let $r > s > 0$. Prove that

$$\frac{M(s)}{s^n} \geq \frac{M(r)}{r^n}.$$

3. Is there an analytic function f that maps $|z| < 1$ into $|z| < 1$ such that $f(\frac{1}{2}) = \frac{2}{3}$, $f(\frac{1}{4}) = \frac{1}{3}$?
4. Suppose u_n is a sequence of harmonic functions on a domain W and suppose the sequence converges uniformly on compact sets to a function u . Prove that u is harmonic.
5. Let $f(z) = \frac{z-a}{1-\bar{a}z}$, where $|a| < 1$. Let $D = \{z : |z| < 1\}$. Prove that

(a)

$$\frac{1}{\pi} \int_D |f'(z)|^2 dx dy = 1.$$

(b)

$$\frac{1}{\pi} \int_D |f'(z)| dx dy = \frac{1-|a|^2}{|a|^2} \log \left(\frac{1}{1-|a|^2} \right).$$

Hint: Use the Poisson integral formula.

6. Let $u(x, y)$, $v(x, y)$ be continuously differentiable as functions of (x, y) in a domain Ω . Let $f(z) = u(z) + iv(z)$. Suppose that for every $z_0 \in \Omega$ there is an r_0 (depending on z_0) such that

$$\int_{|z-z_0|=r} f(z) dz = 0,$$

for all r with $r < r_0$. Prove that f is analytic in Ω . Hint: Show that f satisfies the Cauchy-Riemann equations in Ω .

7. Compute

$$\int_{|z|=4} \frac{\sin z}{z^2} dz.$$

8. Let $D_2 = \{z : |z| < 2\}$ and $I = \{x \in \mathbf{R} : -1 \leq x \leq 1\}$. Find a bounded harmonic function u , defined in $D_2 - I$ such that u does not extend to a harmonic function defined in all of D_2 .

9. Suppose f is analytic on $D = \{|z| < 1\}$ and $f(0) = 0$. Prove that

$$\sum f(z^n)$$

converges uniformly on compact subsets of D .

10. Let a_k be a sequence of distinct complex numbers such that $\sum_{k=1}^{\infty} \frac{1}{|a_k|}$ converges.

Let $A = \{a_k : k = 1, \dots, \infty\}$. Prove that

$$\sum_{k=1}^{\infty} \frac{1}{z - a_k}$$

converges to an analytic function on $\mathbb{C} - A$.

11. Let f and g be entire functions so that satisfy $f^2 + g^2 = 1$. Prove that there is an entire function h so that $f = \cos(h)$, $g = \sin(h)$.

12. Find a function, $h(x, y)$, harmonic in $\{x > 0, y > 0\}$, such that

$$h(x, y) = \begin{cases} 0 & \text{if } 0 < x < 2, y = 0, \\ 1 & \text{if } x > 2, y = 0, \\ 2 & \text{if } x = 0, y > 0 \end{cases}$$

13. Suppose that u is harmonic on all of \mathbb{C} and $u \geq 0$. Prove that u is constant.

14. Suppose f is analytic on $H = \{z = x + iy : y > 0\}$ and suppose $|f(z)| \leq 1$ on H and $f(i) = 0$. Prove

$$|f(z)| \leq \left| \frac{z - i}{z + i} \right|.$$

15. Compute

$$\int_0^{\infty} \frac{1 - \cos x}{x^2} dx.$$

16. Let f be a non-constant analytic function on the connected open set W . Let $Z = \{z : f(z) = 0\}$. Prove that $W - Z$ is connected.

17. Find the radius of convergence of

$$\sum \frac{n^n}{n!} z^{2n}.$$

18. Suppose $f \in \mathcal{O}(0 < |z - a| < \epsilon)$ and that $\operatorname{Re}(f)$ is bounded. Prove that a is a removable singularity.

19. Let f be a non-constant analytic function defined on $\{|z| < 1\}$ such that $\operatorname{Re}(f(z)) \geq 0$.

(a) Prove that $\operatorname{Re}(f(z)) > 0$.

(b) Suppose $f(0) = 1$. Prove that

$$\frac{1 - |z|}{1 + |z|} \leq |f(z)| \leq \frac{1 + |z|}{1 - |z|}.$$

20. Suppose f is analytic on the connected open set W . Let $\{|z - z_0| \leq a\} \subset W$.

(a) Prove that

$$f(z_0) = \frac{1}{\pi a^2} \int_{|z - z_0| \leq a} f(z) dA.$$

(b) Suppose f is not constant on W . Prove that

$$|f(z_0)| < \frac{1}{\pi a^2} \int_{|z - z_0| \leq a} |f(z)| dA \text{ (strict inequality).}$$

21. Suppose f and g are analytic on a connected open set Ω . You might want to use the previous problem on this problem.

(a) If $|f(z)| + |g(z)|$ is constant, then both f and g are constant.

(b) If $|f(z)| + |g(z)|$ assumes a local maximum in Ω , then f and g are constant.

22. Prove that $\sum_1^{\infty} \frac{\sin nz}{2^n}$ represents an analytic function on $|\operatorname{Im}(z)| < \log 2$.

23. (a) Prove that the series

$$\sum_1^{\infty} 2^{-n^2} z^{2^n}$$

converges uniformly on $|z| \leq 1$.

(b) Prove that the radius of convergence of the series is 1.

24. Let f be analytic on the connected open set W . Suppose $\{z : |z - a| \leq d\} \subset W$ and suppose f is real on $\{z : |z - a| = d\}$. Prove that f is constant in W .

25. Gamelin, §VI.2, # 13.

Suppose f has an isolated singularity at a and $\operatorname{Re}(f(z))$ is bounded on $0 < |z - a| < \epsilon$. Prove that the singularity is removable.

Suppose $f(z) = u(z) + iv(z)$ is entire and $|u(z)| > |v(z)|$ for all z . Prove that f is constant.

Suppose f is analytic in $\{0 < |z| < r\}$ for some $r > 0$. Suppose also that $|f(z)| < |z|^{-1+\epsilon}$ in $\{0 < |z| < \delta\}$, where $\epsilon > 0$. Prove that f has a removable singularity at 0.

Let $\sum_{n=-\infty}^{\infty} a_n z^n$ of Laurent series of $\frac{1}{\sin(z)}$ in $|z| < \pi$. Prove that $a_n = 0$ if $n < -1$ and $a_n = 0$ if n is even. Compute a_{-1} and a_1 .

There may be homework problems or example problems from the text on the final. Don't forget previous sample problems.